

INVARIANT RINGS AND QUASIAFFINE QUOTIENTS.

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ABSTRACT. We study Hilbert's fourteenth problem from a geometric point of view. Nagata's celebrated counterexample demonstrates that for an arbitrary group action on a variety the ring of invariant functions need not be isomorphic to the ring of functions of an affine variety. In this paper we will show that nevertheless it is always isomorphic to the ring of functions on a quasi-affine variety.

1. INTRODUCTION

The fourteenth of Hilbert's famous problems ([8]) is the following.

Let K/L and L/k be field extensions, and $A \subset K$ be a finitely generated k -algebra. Does this imply that $A \cap L$ is also a finitely generated k -algebra?

This problem was motivated by the following special case:

Let k be a field and $G \subset GL(n, k)$ a subgroup. Is the ring of invariants $k[x_1, \dots, x_n]^G$ a finitely generated k -algebra?

(This is a special case: Take $K = k(x_1, \dots, x_n)$ and $L = K^G$.)

For reductive groups this is indeed the case. This was already shown by Hilbert. However, for non-reductive groups there is the celebrated counterexample of Nagata [9]). Popov deduced from Nagata's example that for every non-reductive algebraic group G there exists an affine G variety such that the ring of invariants is not finitely generated [13]. In 1990, a new counterexample was found by Roberts [15]. Later, further counterexamples were obtained by Deveney and Finston [4] and by A'Campo-Neuen [1]. Recently, Daigle and Freudenburg constructed examples in dimension 6 and 5 ([3],[6]).

Reformulated in a more geometric fashion, Hilbert's 14th problem ask whether the ring of invariant functions is necessarily isomorphic to the ring of regular functions on some affine variety.

From this point of view it is maybe not too surprising that the answer is negative in general. Quotients of affine varieties by actions of (non-reductive) algebraic groups are often quasi-affine without being affine, and for arbitrary quasi-affine varieties the ring of regular functions is not necessarily finitely generated (see e.g. [11],[12],[14]). Thus even if the ring of invariants is not isomorphic to the ring of regular functions on an affine variety it nevertheless may be isomorphic to the ring of regular functions

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on a quasi-affine variety. The purpose of this note is to demonstrate that this is indeed always the case. Actually we show that the a k -algebra occurs as the ring of invariants for some affine G -variety if and only if it is isomorphic to the algebra of regular functions on some quasi affine variety.

Theorem 1. *Let k be a field and R an integrally closed k -algebra.*

Then the following properties are equivalent:

1. *There exists an irreducible, reduced k -variety V and a subgroup $G \subset \text{Aut}_k(V)$ such that $R \simeq k[V]^G$.*
2. *There exists a quasi-affine irreducible, reduced k -variety V such that $R \simeq k[V]$.*
3. *There exists an affine irreducible, reduced k -variety V and a regular action of $G_a = (k, +)$ on V defined over k such that $R \simeq k[V]^{G_a}$.*

If $\text{char}(k) = 0$, these properties are furthermore equivalent to the following:

4. *There exists a finitely generated, integrally closed k -algebra A and a locally nilpotent derivation D on A such that $R \simeq \ker D$.*

This result is based on the following more general theorem.

Theorem 2. *Let k be a field, V an irreducible, reduced, normal k -variety, and L a subfield of the function field $k(V)$, containing k . Let $R = k[V] \cap L$.*

Then there exists a finitely generated k -subalgebra R_0 of R such that

1. *The quotient fields of R and R_0 coincide.*
2. *For every prime ideal p of height one in R the prime ideal $p \cap R_0$ of R_0 also has height one.*
3. *There is an open k -subvariety $\Omega \subset \text{Spec}(R_0)$ such that $R = k[\Omega]$ (as subsets of $Q(R)$).*

These results can be used to construct some “quasi-affine” quotient for a group action on an algebraic variety.

Theorem 3. *Let k be a field, V an irreducible, reduced, normal k -variety and $G \subset \text{Aut}(V)$.*

Then there exists a quasi-affine k -variety Z and a rational map $\pi : V \rightarrow Z$ such that

1. *The rational map π induces an inclusion $\pi^* : k[Z] \subset k[V]$.*
2. *The image of the pull-back $\pi^*(k[Z])$ coincides with the ring of invariant functions $k[V]^G$.*
3. *For every affine k -variety W and every G -invariant morphism $f : V \rightarrow W$ there exists a morphism $F : Z \rightarrow W$ such that $F \circ \pi$ is a morphism and $f = F \circ \pi$.*

We may also translate our results in the language of category theory (also known as a “general nonsense”) and deduce the following.

Theorem 4. *For a field k let \mathcal{V}_k denote the category whose objects are irreducible reduced normal k -varieties and whose morphisms are those dominant rational maps for which the pull-back of every regular function is again regular. Let \mathcal{Q}_k denote the full sub-category whose objects consist of all quasiasfine such varieties.*

Then for every object $V \in \mathcal{V}_k$ and every subgroup $G \subset \text{Aut}_{\mathcal{V}_k}(V)$ the functor $\text{Mor}_{\mathcal{V}_k}(V, \cdot)^G$ is representable in the category \mathcal{Q}_k .

2. PREPARATIONS

Following ideas of Nagata [9] we employ the notions of *Krull rings* and *ideal transforms* as algebraic tools for our constructions. We give proofs for some basic facts although they are well known. This is for the benefit of being self-contained and because the proofs are so short.

2.1. Krull rings.

Definition 1. *An integral domain R is called a Krull ring if there is a family F of discrete valuations on the quotient field K of R such that $R = \{f \in K : v(f) \geq 0 \ \forall v \in F\}$ and $\{v \in F : v(f) \neq 0\}$ is finite for every $f \in K$.*

For basic facts on Krull rings, see [2][9]. Noetherian integral domains integrally closed in their quotient fields are Krull rings. Intersections of Krull rings inside a fixed field are again Krull rings. For any Krull ring the family of valuations F necessarily contains valuations associated to all prime ideals of height one, on the other hand for a Krull ring F can be chosen as the set of all discrete valuations associated to prime ideals of height one.

Next, we recall that for every k -variety V the ring of functions $k[V]$ is a Krull ring and for every Krull ring R and every group G acting on R by ring automorphisms R^G is again a Krull ring.

Lemma 1. *Let k be a field, V an irreducible, reduced and normal k -variety. Then $k[V]$ is a Krull ring.*

Proof. Let $(U_i)_{i \in I}$ be a collecting of affine k -varieties covering V . For every $i \in I$ the ring $k[U_i]$ is noetherian and integrally closed and therefore a Krull ring. Now $k[V]$ equals the intersection of all the $k[U_i]$ considered as subrings of the function field $k(V)$. Hence $k[V]$ is a Krull ring. \square

Lemma 2. *Let R be a Krull ring and $G \subset \text{Aut}(R)$.*

Then R^G is a Krull ring.

Proof. This is immediate, because R^G is the intersection of two Krull rings, namely R and $Q(R)^G$ where $Q(R)$ denotes the quotient field of R . \square

2.2. The Ideal transform.

Definition 2. Let R be an integral domain, I an ideal. Then the I -transform of R is defined as $S(I, R) = \{x \in K : \exists n : x(I^n) \subset R\}$ where K denotes the quotient field of R .

For Krull rings the ideal transform has particular nice descriptions in geometric as well as in algebraic form. First we explain the geometric description.

Lemma 3. Let R be a Krull ring, I an ideal.

Then $S(I, R)$ equals the ring of regular functions on $\text{Spec } R \setminus V(I)$ (both considered as subsets of the quotient field of R).

Proof. Let K denote the quotient field of R . Let $f \in S(I, R)$ and $p \in X = \text{Spec } R \setminus V(I)$. Since $p \notin V(I)$, the prime ideal p of R does not contain I . Hence there is an element $h \in I \setminus p$. By definition $fh^n \in R$ for some natural number n . Since $h \notin p$, this implies $f \in R_p$.

We will now show the converse. Let f be a regular function on X , i.e., $f \in R_p \forall p \not\supset I$, and let $\Lambda = \{v \in F : v(f) > 0\}$. For $v \in \Lambda$ consider the associate prime ideal $p_v = \{g \in R : v(g) < 0\}$. Now $v(f) > 0$ implies that $f \notin R_{p_v}$, hence $I \subset p_v$. Thus $v(x) < 0$ for all $x \in I$ and $v \in \Lambda$. Since Λ is finite, it follows that $fI^n \subset R$ for n sufficiently large, i.e. $f \in S(I, R)$. \square

Next we come to the algebraic description of the ideal transform.

Lemma 4. Let R be a Krull ring, K its quotient field, F the set of all discrete valuations corresponding to prime ideals of height one in R and I an ideal in R . Then $S(I, R) = \{x \in K : v(x) \leq 0 \ \forall v \in F'\}$ with $F' = \{v \in F : I \not\subset p_v\}$.

Proof. Let $x \in S(I, R)$ and $v \in F'$. Then there is an element $f \in I$ such that $v(f) = 0$. Now $xI^n \subset R$ implies $v(x) \leq 0$. Conversely, assume x is an element in K such that $v(x) \leq 0$ for all $v \in F'$. Let $\Lambda = \{v \in F : v(x) > 0\}$. Then Λ is finite and $I \subset p_v$ for all $v \in \Lambda$. This implies $xI^n \subset R$ for n sufficiently large. \square

We will need the fact that for prime ideals of height one the ideal transform is non-trivial.

Lemma 5. Let R be a Krull ring, p a prime ideal of height one. Then $S(p, R) \neq R$.

Proof. Let K be the quotient field of R , $R = \{x \in K : v(x) \leq 0 \ \forall v \in F\}$, $p = \{x \in R : v_0(x) < 0\}$. Choose $x \in P$ and define $\Lambda = \{v \in F : v(1/x) > 0\} \setminus \{v_0\}$. Now Λ is finite, and p_v being a prime ideal of height one for every v implies that p_v is not contained in p for any $v \in \Lambda$. Hence there is an element $y \in R$ with $y \notin p$ but $v(y) < 0$ for all $v \in \Lambda$. Then $y^n x \in S(p, R) \setminus R$ for n sufficiently large. \square

2.3. In the situation in which we are interested prime ideals of height one are determined by their zero sets.

Lemma 6. *Let k be a field, V an integral, normal k -variety, R a k -sub algebra of $k[V]$ such that R is a Krull ring and $R = Q(R) \cap k[V]$.*

Let p be a prime ideal of height one in R . Then $p = I(Z(p)) \cap R$, where $I(Z(p))$ denotes the set of all $f \in k[V]$ vanishing on the zero set of p .

Moreover, $Z(p)$ contains an irreducible component Z_0 such that $I(Z_0) = p$.

Proof. Let v denote the discrete valuation corresponding to p . By (*) there is an element $x \in Q(R) \setminus R$ with $v(x) > 0$ and a number N such that $xp^N \subset R$. Thus x defines a rational function on V whose poles are contained in $Z(p)$. Now assume $g \in I(Z(p))$ but $g \notin p$. This would imply that $v(g) = 0$ and g vanishes on $Z(p)$. Since the poles of x are contained in $Z(p)$, it follows that $g^m x \in k[V]$ for m large enough. On the other hand $v(g^m x) = v(x) > 0$ implies $g^m x \notin R$. Thus the assumption $I(Z(p)) \neq p$ yields a contradiction to the requirement that $R = Q(R) \cap k[V]$.

For the final statement, let $(Z_i)_{i \in I}$ denote the irreducible components of $Z(p)$. Then $I(Z(p)) = \cap_i I(Z_i)$. Since both $I(Z(p))$ and all of the $I(Z_i)$ are prime ideals, it follows that there exists an $i \in I$ such that $I(Z_i) = I(Z(p)) = p$. \square

Lemma 7. *Let k be a field and V a k -variety. For every k -algebra $R \subset k[V]$ let \sim_R denote the equivalence relation on the geometric (\bar{k} -rational) points of V given by $x \sim_R y$ iff $f(x) = f(y)$ for all $f \in R$.*

Then for every k -sub algebra $R \subset k[V]$ there exists a finitely generated k -algebra $R_0 \subset R$ such that \sim_R coincides with \sim_{R_0} .

Furthermore R_0 can be chosen as a Krull ring, if R is a Krull ring.

Proof. The equivalence relation \sim_R defines a subset $E_R \subset V \times V$ via

$$(*) \quad E_R = \{(x, y) \in V \times V : x \sim_R y\} = \{(x, y) : f(x) = f(y) \forall f \in R\}$$

Then E_R is the k -sub variety defined by the radical of the ideal of $k[V \times V]$ generated by $\pi_1^* f - \pi_2^* f$ with f running through R . If $R' \subset R$ are two k -sub algebras of $k[V]$, then $E_R \subset E_{R'}$. Since $V \times V$ is noetherian it follows that for any k -sub algebra $R \subset k[V]$ there exists a finitely generated k -sub algebra $R_0 \subset R$ such that $E_R = E_{R_0}$.

Finally, if R is a Krull ring, then R is integrally closed in its quotient field. Hence the integral closure of R_0 is again contained in R . Furthermore the integral closure of R_0 is again finitely generated as a k -algebra. Thus we may assume that R_0 is integrally closed. But integrally closed finitely generated k -algebras are Krull rings. \square

Let us now fix some key assumptions.

Key Assumptions. *In the sequel, k is a field, V an irreducible, reduced and normal k -variety, $R \subset k[V]$ a k -sub algebra such that $Q(R) \cap k[V] = R$, R_0 is a finitely generated k -algebra such that $R_0 \subset R$, $Q(R_0) = Q(R)$, R_0 is integrally closed in $Q(R_0) = Q(R)$ and $E_R = E_{R_0}$ where E_R is defined as in (*) above.*

We will prove the statements of theorem 2 hold for such a choice of R_0 .

Lemma 8. *Under the key assumptions $\text{height}(p \cap R_0) = 1$ for every prime ideal p of height one in R .*

Proof. Let $W = \text{Spec}(R_0)$, this is an affine k -variety. The prime ideals of height one in R_0 correspond to the irreducible hypersurfaces in W . Let $\tau : V \rightarrow W$ denote the morphism induced by $R_0 \subset k[V]$. Let p be a prime ideal of height one in R . We have seen above that there is an irreducible subvariety $Z \subset V$ such that $I(Z) \cap R = p$.

From $E_R = E_{R_0}$ we infer that there exists an irreducible subvariety $Y \subset Z$ such that $\tau|_Y : Y \rightarrow W$ is generically quasi-finite and Z is the smallest E_R -saturated subvariety containing Y . Now let X be an irreducible subvariety of V such that $\dim(X) = \dim(Y) + 1$ and $X \not\subset Z(p)$. Let $I = I(X) \cap R$. Then $p \neq I$. On the other hand $I \subset I(Y)$ and $I(Y) \cap R = I(Z) \cap R = p$. Thus $\text{height}(p) = 1$ implies $I = \{0\}$. It follows that $\tau(X)$ must be Zariski-dense in $W = \text{Spec}(R_0)$. Since $\dim(X) = \dim(Y) + 1$ it now follows from $\tau|_Y$ being generically quasi-finite that $\overline{\tau(Y)}$ either is a hypersurface or equals W . The latter is excluded since $p \neq \{0\}$ and $Y \subset Z(p)$. Thus $\overline{\tau(Y)} = \overline{\tau(Z)}$ has to be a hypersurface implying that $\text{height}(p \cap R_0) = 1$. \square

Lemma 9. *Under the key assumptions for any two distinct prime ideals of height one $p_1, p_2 \subset R$ we have $p_1 \cap R_0 \neq p_2 \cap R_0$.*

Proof. This is an immediate consequence of $E_R = E_{R_0}$ and $I(Z(p_i)) \cap R = p_i$. \square

Corollary 1. *Let F resp. F_0 denote the set of discrete valuations of $K = Q(R)$ corresponding to prime ideals of height one in R resp. R_0 . Then $F \subset F_0$.*

Lemma 10. *The set $F_0 \setminus F$ is finite.*

Proof. Let $v \in F_0 \setminus F$, $H_0 \subset W = \text{Spec}(R_0)$ the corresponding hypersurface and $H = \tau^{-1}(H_0)$. We claim that $\tau(H)$ is not Zariski-dense in H_0 . Indeed, if it were Zariski-dense, there would exist an irreducible subvariety H' with $\overline{\tau(H')} = H$. But this implies $I(H') \cap R_0 = I(H_0)$. Now let p be a prime ideal of height one contained in $I(H')$ (such an ideal exists, because $I(H')$ is a prime ideal and R is a Krull ring). Then $p \cap R_0$ is a prime ideal of height one. Since $I(H_0)$ is of height one, it follows that $I(H_0) = p \cap R_0$ contrary to our assumption $v \notin F$. Thus H is a hypersurface in W with $\tau(\tau^{-1}(H))$ not being dense in H . Since $\tau : V \rightarrow W$ is dominant, there are only finitely many such hypersurfaces in W . \square

3. PROOFS OF THE THEOREMS

3.1. Proof of theorem 2.

Proof. Let k , V , L and R be as in the theorem. Note that L is a finitely generated field extension of k , because $k(V)/k$ is finitely generated and $L \subset k(V)$.

By lemma 7 there is a finitely generated k -algebra R_1 with $R_1 \subset R$, $E_R = E_{R_1}$ and R_1 being a Krull ring. We may adjoin finitely many further elements of R to R_1 and thereby assume that the quotient fields of R and R_1 coincide. Then we choose R_0 as the integral closure of R_1 in L . Since R is integrally closed, we have $R_0 \subset R$. Furthermore $E_R = E_{R_0}$ and R_1 is again finitely generated, because it is the integral closure of a finitely generated k -algebra.

Thus R_0 fulfills the “Key Assumptions”. Statement (1) of the theorem is clear, and statement (2) follows from lemma 8.

Next we define F and F_0 as in the corollary above. By lemma 10 the difference set $F_0 \setminus F$ is finite. Hence we may define an ideal of R_0 by

$$I = \prod_{v \in \{F_0 \setminus F\}} p_v$$

with $p_v = \{x \in R_0 : v(x) < 0\}$.

Since p_μ is of height one for every $\mu \in F_0$, it is clear that $I \not\subset p_v$ for $v \in F$. Therefore $R = S(I, R_0)$ by lemma 4.

Finally, statement (3) of the theorem follows with the aid of lemma 3. \square

Before starting the proof of theorem 3 we need the subsequent lemma.

Lemma 11. *Let k be a field, V a k -variety and W a quasi-affine k -variety.*

*Then there is a one-to-one correspondence between k -algebra homomorphisms $\phi : k[W] \rightarrow k[V]$ and rational maps $f : V \rightarrow W$ with $f^*k[W] \subset k[V]$.*

Proof. Given a k -algebra homomorphism $\phi : k[W] \rightarrow k[V]$, choose a finitely generated k -sub algebra $A \subset k[W]$ such that the quotient fields of $k[W]$ and A coincide. The restriction of ϕ yields a k -morphism F from V to $Z = \text{Spec}(A)$ and the inclusion $A \subset k[W]$ yields a birational morphism $\tau : W \rightarrow Z$. Now $\tau^{-1} \circ F$ is the desired rational map. \square

3.2. Proof of theorem 3.

Proof. We apply theorem 2 with $L = k(V)^G$ and set $Z = \Omega$. There is an inclusion $k[Z] = k[\Omega] = R \subset k[V]$. By the preceding lemma this induces a rational map $\pi : V \rightarrow Z$ with $\pi^*k[Z] \subset k[V]$. Since $R = k[V] \cap L = k[V]^G$, it follows that $\pi^*k[Z] = k[V]^G$.

Finally note that for every affine k -variety W and every G -invariant morphism $f : V \rightarrow W$ we obtain an inclusion $f^*k[W] \subset k[V]^G \simeq k[Z]$ which implies that there exists a morphism $F : Z \rightarrow W$ such that $f = F \circ \pi$. \square

3.3. Proof of theorem 1.

Proof. The implication (1) \implies (2) follows from theorem 3. (3) \implies (1) is trivial and (2) \implies (3) follows from the proposition below.

Finally, (3) \iff (4) for the case of characteristic zero is implied by the well-known correspondence between locally nilpotent derivations and G_a -actions on affine varieties in characteristic zero. \square

Proposition 1. *Let k be a field and let V be a normal quasi-affine k -variety.*

Then there exists a normal affine k -variety W and a regular action of the additive group $G = G_a$ defined over k on W such that $k[V] \simeq k[W]^G$.

Proof. Let $V \hookrightarrow Y$ be an open embedding in a normal affine k -variety Y , and let $S = Y \setminus V$. Let D denote the union of codimension 1-components of S and choose a regular function f_1 on Y such that D is contained in the zero set of f_1 . Then choose a regular function f_2 on Y such that f_2 vanishes on D , but does not vanish on any irreducible component of $Z(f_1) \setminus D$. If $\text{char}(k) = p > 0$, we replace f_i by $f_i^{p^N}$ for a sufficiently large N . In this way we may assume that both functions f_i are defined over a finite Galois extension k'/k with Galois group Γ . Now we may replace f_i by $\prod_{\sigma \in \Gamma} \sigma f_i$. Therefore we may assume that both f_i are defined over k . We obtain a k -morphism $f : Y \rightarrow \mathbb{A}^2$. By construction D is the union of codimension 1-components of $E = f^{-1}\{(0, 0)\}$. Since regular functions extend through subvarieties of codimension at least 2 on normal varieties, it follows that

$$k[V] \simeq k[Y \setminus D] \simeq k[\Omega]$$

for $\Omega = Y \setminus E$.

Next we consider

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

and the natural projection $\pi : S \rightarrow \mathbb{A}^2$ given by $\pi : (a, b, c, d) \mapsto (a, b)$. This realizes $\mathbb{A}^2 \setminus \{(0, 0)\}$ as the quotient of S by the G_a -action given by

$$t : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b + ta \\ c & d + tc \end{pmatrix}$$

Now the fiber product $W = Y \times_{\mathbb{A}^2} S$ is an affine variety and $W \simeq \Omega \times_{\mathbb{A}^2 \setminus \{(0, 0)\}} S$, because the image of S in \mathbb{A}^2 is contained in $\mathbb{A}^2 \setminus \{(0, 0)\}$ and $\Omega = f^{-1}(\mathbb{A}^2 \setminus \{(0, 0)\})$. The G_a -action on S induces a G_a -action on the fibered product W and evidently $k[\Omega] \simeq k[W]^{G_a}$. \square

3.4. Proof of theorem 4.

Proof. Let $X \in \mathcal{V}_k$ and $G \subset \text{Aut}_{\mathcal{V}_k}(X)$. Each element in G is a birational self-map of X such that the induced field automorphism of $k(X)$ stabilizes $k[X]$. In particular $G \hookrightarrow \text{Aut}(k[X])$. Due to theorem 2 there exists an object $\Omega \in \mathcal{Q}_k$ and a \mathcal{V}_k -morphism $\pi : X \rightarrow \Omega$ such that $k[\Omega] \xrightarrow{\sim} k[X]^G$.

Consider now an object $W \in \mathcal{Q}_k$ with a G -invariant \mathcal{V}_k -morphism $f : X \rightarrow W$. Then f is dominant rational map with $f^*(k[W]) \subset k[X]^G$. Thus we obtain a k -algebra homomorphism $f^* : k[W] \rightarrow k[\Omega] = k[X]^G$, which by lemma 11 induces a \mathcal{Q}_k -morphism from Ω to W .

Therefore $Mor_{\mathcal{V}_k}(X, W)^G \simeq Mor_{\mathcal{Q}_k}(\Omega, W)$. \square

4. AN EXAMPLE

Let k be a field of characteristic zero. In [3] Daigle and Freudenburg gave an example of a locally nilpotent derivation D of $k[\mathbb{A}^5]$ such that $\ker D$ is not finitely generated. This is the lowest-dimensional example known today. In coordinates x, s, t, u, v the derivation D can be written as

$$D = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v}$$

The associated group action of the additive group G_a is given by

$$\mu(r) : (x, s, t, u, v) \mapsto (x, s + rx^3, t + rs + \frac{r^2}{2}x^3, u + rt + \frac{r^2}{2}s + \frac{r^3}{6}x^3, v + rx^2)$$

The action is free outside the set of fixed points

$$(\mathbb{A}^5)^\mu = \{(0, 0, 0, v, u) : u, v \in k\}$$

An explicit calculation shows that the following regular functions on \mathbb{A}^5 are invariant:

$$\begin{aligned} \phi_1 &= x \\ \phi_2 &= 2x^3t - s^2 \\ \phi_3 &= 3x^6u - 3x^3ts + s^3 \\ \phi_4 &= xv - s \\ \phi_5 &= x^2ts - s^2v + 2x^3tv - 3x^5u = (\phi_2\phi_4 - \phi_3)/\phi_1 \\ \phi_6 &= -18x^3tsu + 9x^6u^2 + 8x^3t^3 + 6s^3u - 3x^6t^2s^2 = (\phi_2^3 + \phi_3^2)/\phi_1^6 \end{aligned}$$

Further explicit calculations yields the following:

Lemma 12. *Let $V = \{w = (w_1, \dots, w_6) \in \mathbb{A}^6 : w_5w_1 = w_2w_4 - w_3, w_6w_1^6 = w_2^3 + w_3^2\}$. Then V is an affine subvariety of \mathbb{A}^6 with $Sing V = \{w \in \mathbb{A}^6 : w_1 = w_2 = w_3 = 0\}$. $Sing V$ is a Weil divisor of V .*

The regular functions $(\phi_i)_{i=1..6}$ defined above give an invariant morphism $\phi : \mathbb{A}^5 \rightarrow V$ such that

1. ϕ has rank 4 outside $E = \{x = s = 0\}$.
2. ϕ maps $\mathbb{A}^5 \setminus E$ surjectively on $V \setminus Sing V$.
3. For every $p \in V \setminus Sing V$ the fiber $\phi^{-1}(p)$ coincides with a G_a -orbit.

It follows that

$$k[\mathbb{A}^5]^\mu = k[\mathbb{A}^5 \setminus E]^\mu \simeq k[V \setminus \text{Sing } V]$$

Thus the ring of invariants is indeed isomorphic to the ring of regular functions of a quasi affine variety, namely $V \setminus \text{Sing } V$.

5. APPENDIX

Naturally one would prefer having a regular morphism in (*) instead of having merely a rational map. One might hope for a general result implying that this map is automatically regular, and endeavor to prove a statement like the following.

Falsity. *Let V , W be affine varieties, $f : V \rightarrow W$ be a regular map, $H \subset W$ a hypersurface and assume that $f^*(k[W \setminus H]) \subset k[V]$. Then $f(V) \subset W \setminus H$ (implying that one has a regular morphism from V to $W \setminus H$ and not merely a rational map.)*

But this is wrong:

Example. *Let $V = \{(x_1, x_2, x_3) \in \mathbb{A}^3 : x_1 \neq 0\}$, $W = \{(z_1, z_2, z_3, z_4) \in \mathbb{A}^4 : z_1 z_4 = z_2 z_3\}$, $H = \{(z_1, z_2, z_3, z_4) : z_1 = z_2 = 0\}$ and $f : V \rightarrow W$ given by $f(x_1, \dots, x_3) = (x_1 x_2, x_1 x_3, x_2, x_3)$. Then H is a hypersurface in W , but $f^{-1}(H) = \{(x_1, 0, 0) : x_1 \neq 0\}$ is a curve, and therefore has codimension two in V . As a consequence $f^*k[W \setminus H] \subset k[V \setminus f^{-1}(H)] = k[V]$, although $f(V) \cap H = \{(0, 0, 0, 0)\} \neq \emptyset$.*

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